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R. Casalbuoni
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P. Verrucchi

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A HISTORY OF FEYNMAN'S SUM OVER HISTORIES IN QUANTUM MECHANICS

AKIRA INOMATA
Department of Physics, State University of New York at Albany, Albany, New York 12222, U.S.A.

GEORG JUNKER
Institut für Theoretische Physik, Universität Erlangen-Nürnberg, Staudtstr. 7, D-91058 Erlangen, Germany

A history of Feynman's sum over histories is presented in brief. A focus is placed on the progress of path-integration techniques for exactly path-integrable problems in quantum mechanics.

1 Classification of Path Integrals

Exact calculations of Feynman's path integrals (defined on a time lattice) are mainly based on recurrence integral formulas in which the convolution of two functions having a common feature retains the same feature. Therefore, exactly soluble path integrals in quantum mechanics may be classified by their recurrence integral formula used in the calculation. According to this classification, there are three types of path integrals: (i) *Gaussian path integrals*, (ii) *Legendrean path integrals*, and (iii) *Besselian path integrals*. The Gaussian path integrals are calculated by the well-known convolution of two Gaussian functions which produces recurrently another Gaussian function. Path integrals of this type have been widely used in semiclassical approaches, coherent-state path integrals, applications in statistical physics, field theory and many others areas. The Legendrean path integrals are based on the convolution integral for zonal spherical functions (generalized Legendre functions), which are particular matrix elements of unitary irreducible representations of Lie groups. An elementary type of the Legendrean path integrals appears in the angular path integral in 3-dimensional polar coordinates. More generally, path integrals of this type are useful for systems with certain symmetries of kinematical or dynamical origin. For details see the review [523]. The Besselian path integrals are based on Weber's integral formula for Bessel functions [771] or the group composition law (convolution) for particular unitary representations of an element of $SU(1,1)$ in a continuous base. Radial path integrals are of the Besselian type and may be associated with a certain dynamical group or spectrum generating algebra [523].

The books of Feynman and Hibbs [340] and of Schulman [828] discuss mainly path integrals of the Gaussian type which are undoubtedly most important in applications. However, to understand the complete feature of Feynman's path integral, we cannot ignore the non-Gaussian aspects. Indeed, the non-Gaussian features have been found essential in studying exactly path-integrable systems. The history of those non-Gaussian path integrals began with the polar-coordinate formulation of path integrals.

2 Polar Coordinate Formulation and Harmonic analysis

In the early 1950's, Ozaki (lecture notes, Kyushu University, 1955, unpublished) started with a short-time action for a free particle written in cartesian coordinates and transformed it to the polar form. Then he performed angular path integration but failed in carrying out explicitly the radial path integral. In 1964, Edwards and Gulyaev [291] formulated the free particle propagator in polar coordinates in a way similar to that of Ozaki. They have completed the radial path integration for the free particle. In 1969, Peak and Inomata [771] calculated explicitly the radial path integral for the harmonic oscillator (in an inverse-square potential). This opened the direction of the Besselian path integral which became an important base for the study of exactly path-integrable systems [528].

In addition it became clear that group theoretical methods, most notably harmonic analysis, are not only elegant but also powerful tools in path integration. In 1968 Schulman [826] discussed spin with the path integral of a rigid rotor in terms of $SU(2)$ representations. In 1970 Dowker [262] discussed exactness of semiclassical results on compact groups such as $SU(n)$. Marinov and Terentyev [679], in 1979, used harmonic analysis for path integration on $SO(n+1)/SO(n)$. In 1987 Böhm and Junker [104] generalized this to formulate path integrals on symmetric spaces including $SU(1,1)$ and $SO(n,1)/SO(n)$, and solved chiefly Legendrean-type path integrals by harmonic analysis. In 1991 a group theoretical treatment of the radial path integrals was made on the basis of $SU(1,1)$ by the present authors [523].

3 The Hydrogen-Atom Problem

Before 1979 the list of exactly path-integrable systems was very short. It is a curious fact that Feynman's path integral could not reproduce the exact solution for the hydrogen atom which once symbolized the success of Schrödinger's wave mechanics. If Feynman's approach is equivalent to Schrödinger's, then it should be able to solve the very standard problem in quantum mechanics. Since the path integral involving the Coulomb potential was not directly integrable, Gutzwiller [749] treated it semiclassically on the basis of his famous trace formula, yielding the exact energy spectrum, whereas Goovaerts

and Devreese [408] made a perturbation calculation and obtained the exact s-wave spectrum.

In 1979, however, Duru and Kleinert [279] made a breakthrough in solving the hydrogen problem by path integration without approximations. They applied the Kustaanheimo-Stiefel (KS) space-time transformation to the path integral. The KS transformation, which consists of a coordinate map from $\mathbb{R}^3$ to $\mathbb{R}^4$ and a path-dependent time transformation, converted the path integral for the hydrogen atom into the exactly soluble path integral for an isotropic harmonic oscillator in four dimensions.

4 Other Exactly Path-Integrable Systems

Although the KS transformation is limited in application, its success in the hydrogen atom suggested that (i) a local time transformation may be used in combination with a nonlinear coordinate transformation in order to reduce a non-soluble path integral to a soluble path integral; and (ii) the dimension of the configuration space for path integration may be extended so as to treat a dynamical symmetry of the system in question as a kinematical one. By these techniques, nearly all the exactly soluble problems by Schrödinger's equation became path-integrable. The local time transformation was taken only as a formal trick in the beginning, but its use was justified for the Wiener-type path integrals (Feynman path integrals in Euclidean time) by Blanchard and Sirugue [97], Young and DeWitt-Morette [943], Castrigiano and Stärk [152], and Fischer, Leschke and Müller [343,344].

Finally we wish to remark that there are two types of time transformations applicable to path integration. One is integrable (globally meaningful), whereas the other is nonintegrable (only locally meaningful). The KS time transformation is of the latter case. A remarkable example involving an integrable time transformation is the one considered by Alfaro, Fubini and Furlen and by Jackiw, which converts a harmonic oscillator into a free particle [128,552 and references therein].

References

There are a number of textbooks on Feynman's path integral. The most recent one is C. Grosche and F. Steiner, "Handbook of Feynman Path Integrals", (Springer, Berlin, 1998), which has an extensive list of references (pp. 368-423) and textbooks (pp.21-22). Specifically the book [528] in the list on page 21 discusses in detail several techniques used for exact path integration in quantum mechanics. In the present article, for convenience, we adopt the reference numbers of the book of Grosche and Steiner; the numbers given in square brackets in the text refer to the corresponding reference numbers.